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LETTER TO THE EDITOR

Stochastic equations for fields in complex manifolds

Z Haba†

Research Centre Bielefeld-Bochum-Stochastics, Bielefeld University, D-4800 Bielefeld, FRG

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Abstract. The construction of the (Euclidean) quantum mechanics on a manifold are generalised to two-dimensional models of quantum field theory. Partial differential equations for Euclidean fields are obtained, which are random perturbations of instanton equations.

Stochastic equations have been derived recently [1, 2] (see also earlier papers [3-5]) for some models of quantum field theory. In this letter stochastic equations are obtained for a larger class of two-dimensional models (detailed study of these equations as well as stochastic equations in higher dimensions in models with instantons will appear later [6]). The stochastic equations on a finite-dimensional manifold constitute a classical part of probability [7]. We derive stochastic equations for fields, treated as coordinates of an infinitely dimensional manifold, through a direct generalisation of the finite-dimensional case (see also [5, 8]).

(1) There is a direct relation between the Hamiltonian H as a second-order elliptic differential operator defined on a Riemannian manifold M and the stochastic process $\xi(t)$ on M

$$(\mathrm{e}^{-tH}f)(x)=E[f(\xi_x(t))] \tag{1}$$

where $\xi_x(t)$ is a solution (with the initial condition $\xi(0)=x$) of the stochastic equation

$$\dot{\xi}=\beta(\xi)+\dot{\eta}. \tag{2}$$

η is the Brownian motion on M [7] ($\dot{\eta}=\mathrm{d}\eta/\mathrm{d}t$), which fulfils the equation

$$\dot{\eta}_\mu=e_{\mu a}(\eta)\dot{b}^a+\tfrac{1}{2}\Gamma_\mu(\eta) \tag{3}$$

here $e_{\mu a}$ is the tetrad ($e_{\mu a}e_{\nu a}=g_{\mu\nu}$), Γ is the Christoffel symbol and $\dot{b}$ is the white noise, i.e.

$$E[\dot{b}^r(t)\dot{b}^s(t')]=\delta^{rs}\delta(t-t'). \tag{4}$$

If $\beta=0$, then $H_0=-\frac{1}{2}\Delta_M$, where Δ_M is the Laplace-Beltrami operator on M. The formal path-space measure for H_0 in (1) has the form (it makes sense on the lattice)

$$\mathrm{d}\mu_0(\eta)=\mathrm{d}\eta(\det g)^{1/2}\exp\left(-\tfrac{1}{2}\int g_{\mu\nu}(\eta)\dot{\eta}^\mu\dot{\eta}^\nu\right). \tag{5}$$

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Now, the functional measure corresponding to the process ξ (2) can be expressed by μ_0 (this is the Cameron-Martin-Girsanov formula [7]; see also [9]):

$$d\mu = d\mu_0(\eta)\rho(\eta) = d\mu_0(\eta)\exp\left(-\tfrac{1}{2}\int (\eta)\beta_\mu(\eta) + \int \beta^\mu(\eta)e_{\mu a}(\eta)\dot{b}^a\right). \tag{6}$$

The Feynman–Kac formula for the potential $U = \frac{1}{2}(\nabla V)^2 - \frac{1}{2}\Delta V$ (then $\beta = -\nabla V$) follows from (6) and from the Ito formula $\dot{f} = \partial_a f\dot{b}^a + \frac{1}{2}\Delta f$.

A formal derivation of the CMG formula (6) will explain its meaning and eventual approximation scheme for computations

$$\begin{aligned} E[F(\xi(b))] &= \int d\mu(\xi)F(\xi) = \int d\dot{b}\,\exp\left(-\tfrac{1}{2}\int \dot{b}^2F(\xi(b))\right) \\ &= \int d\dot{\zeta}\,d\dot{b}\,\exp\left(-\tfrac{1}{2}\int \dot{b}^2\delta(\dot{\zeta}+\dot{\xi}(b))F(\zeta)\right) \end{aligned} \tag{7}$$

whereas

$$\begin{aligned} E[F(\eta(b))\rho(\eta(b))] &= \int d\mu_0(\eta)\rho(\eta)F(\eta) \\ &= \int d\dot{b}\,\exp\left(-\tfrac{1}{2}\int \dot{b}^2\rho(\eta(b))F(\eta(b))\right) \\ &= \int d\dot{\zeta}\,d\dot{b}\,\exp\left(-\tfrac{1}{2}\int (\dot{b}-e\beta(\zeta))^2F(\zeta)\delta(\dot{\zeta}-\dot{\eta}(b))\right) \\ &= \int d\dot{\zeta}\,d\dot{b}'\,\exp\left(-\tfrac{1}{2}\int \dot{b}'^2F(\zeta)\delta(\dot{\zeta}-\beta-\dot{\eta}(b'))\right). \end{aligned} \tag{8}$$

Hence, the expressions (7) and (8) coincide.

(2) As the simplest example consider a process $\varphi = \varphi_1 + i\varphi_2$ on the complex plane $\mathbb{C}$ fulfilling the equation

$$\dot{\varphi} = \overline{V(\varphi)} + \dot{b} \tag{9}$$

where $b = b_1 + ib_2$ is the complex noise and (by assumption) $V(\varphi)$ is a holomorphic function. It can be seen from the formula (6) that V and $e^{i\alpha}V$ describe the same process, because b is invariant under such rotations.

Next, consider the Kähler manifolds. For these complex manifolds the Christoffel symbol Γ in (3) vanishes in complex coordinates. So, only the tetrad remains in the stochastic equation. As an example, the CP^n manifold [10] in the complex coordinates $(w^\alpha, \overline{w^\alpha})$ has the metric

$$g_{\mu\bar{\nu}}(w) = e_{\mu a}(w)\overline{e_{\nu a}(w)} = \tfrac{1}{4}(1+w\bar{w})^{-2}(\delta_{\mu\nu}(1+w\bar{w}) - \bar{w}_\mu w_\nu) \tag{10}$$

where

$$e^{\mu a}(w) = \tfrac{1}{2}(1+w\bar{w})^{1/2}(\delta^{\mu a} - w^\mu\bar{w}^a/w\bar{w}) + \tfrac{1}{2}(1+w\bar{w})w^\mu\bar{w}^a/w\bar{w} \tag{11}$$

and $w\bar{w} = \Sigma\, w^a\bar{w}^a$.

The stochastic equation (3) takes the form

$$\dot{w}_\alpha = e_{\alpha a}(w)\dot{b}^a. \tag{12}$$

CP^n is a compact manifold; therefore the Hamiltonian has a discrete spectrum and the correlation functions of w_t show an exponential decay in time.

(3) A random field $\xi(t, x) = \xi_t(x)$ can be considered as a random curve on a manifold $\mathscr{F}(R, M)$ of maps ξ: $R \to M$, i.e. as a stochastic process in $\mathscr{F}$. $\mathscr{F}$ is a Riemannian Hilbert manifold modelled on $L^2(R)$ [11] with the scalar product for $v \in (T\mathscr{F})_q$ (the tangent space at $q \in \mathscr{F}$) defined by

$$(v, v') = \int \mathrm{d}x (v(x), v'(x))_{q(x)} \tag{13}$$

where (,) is the Riemannian structure in M.

We could consider (3) as a stochastic equation in $\mathscr{F}$ (see [8] for a theory of such equations) with the two-dimensional white noise $\dot{b}(t, x)$

$$E[\dot{b}(t, x)\dot{b}(t', x')] = \delta(t - t')\delta(x - x').$$

However, such an equation would not be Euclidean invariant. Equation (3) will be modified by a proper choice of the drift term β in equation (2). We determine β from the following rules: (i) the solutions of the modified equation should stay on M; (ii) the stochastic equation should remain invariant under space translations $x \to x + a$; and (iii) the exponential decay in time of expectation values of functionals invariant under space translations should be preserved.

The requirements (i)-(iii) will be fulfilled if the drift β is equal to the *Killing vector* K corresponding to the translational isometry of the Riemannian metric (13) (the role of a Killing vector in a related context has been emphasised in [12] and [13]). It is easy to see that the Killing vector K for $\varphi \in \mathbb{C}((v, v') = \int \mathrm{d}x\overline{v(x)}v'(x))$ has the form to a constant) $K = -\mathrm{i}\partial_x\varphi$. With such a choice of β, equation (9) reads ($\bar{\partial}_z = \partial_t + \mathrm{i}\partial_x$)

$$\bar{\partial}_z\varphi = \overline{V(\varphi)} + \dot{b}. \tag{14}$$

This is the equation discussed in [1] and [2]. Equation (14) is obviously invariant under translations $z \to z + c$. Under Euclidean rotations $z \to \mathrm{e}^{\mathrm{i}\alpha}z$, hence in (14) $V \to \mathrm{e}^{\mathrm{i}\alpha}V$ and $b \to \mathrm{e}^{-\mathrm{i}\alpha}b$. As mentioned before solutions of (14) are invariant under such transformations.

For the CP^n model in the complex coordinates $K^\alpha = -\mathrm{i}\partial_x w^\alpha$. Hence, (2) with the drift added to (12) has the form

$$\bar{\partial}_z w^\alpha = e^\alpha{}_a(w)\dot{b}^a. \tag{15}$$

This equation is explicitly invariant under the Euclidean group. Moreover, it is invariant under arbitrary holomorphic transformations $\zeta = \zeta(z)$, because the white noise $\dot{b}$ is invariant under the transformation $\dot{b}(z) \to \overline{\partial\zeta/\partial z}\, \dot{b}(\zeta)$. Note that if the noise $\dot{b}$ is switched off (there is $(\hbar)^{1/2}$ in front of b), then (15) goes over into the *instanton* equation.

(4) Consider now an Abelian Higgs model. The configuration space $\mathscr{F}$ of the Higgs model can be considered as a set $(\tilde{A}, \tilde{\varphi})$ of equivalence classes with respect to the gauge transformations. The metric (13) is invariant under the translation $\varphi(x) \to \exp(\mathrm{i}A_1(x)\mathrm{d}x)\, \varphi(x + \mathrm{d}x)$ of the fibre. The generator K of this isometry is equal to $-\mathrm{i}(\partial_x - \mathrm{i}A_1)\varphi$. In the temporal gauge the configuration space $\mathscr{F}$ is described by A_1 and φ. We write down the stochastic equation in this gauge according to the rules (i)-(iii). The stochastic equation transformed to an arbitrary gauge has the form

$$\begin{aligned}(\partial_t - \mathrm{i}A_0)\varphi &= -\mathrm{i}(\partial_x - \mathrm{i}A_1)\varphi + \dot{b} \\ \partial_t A_1 - \partial_x A_0 &= V(|\varphi|^2) + \dot{\eta}_1.\end{aligned} \tag{16}$$

The Lorentz gauge can be imposed as an auxiliary stochastic equation

$$\partial_t A_0 + \partial_x A_1 = \dot{\eta}_0.$$

Then, (16) can be expressed in a complex form

$$\bar{\partial}_z \varphi = \mathrm{i} A\varphi + \dot{b} \qquad \partial_z A = \mathrm{i}\, V(|\varphi|^2) + \dot{\eta} \tag{17}$$

where $\partial_z = \partial_t - \mathrm{i}\partial_x$, $A = A_0 + \mathrm{i}A_1$ and $\eta = \eta_0 + \mathrm{i}\eta_1$.

Equations (16) and (17) are invariant under the Euclidean group if φ transforms as a scalar and A as a vector under rotations. With

$$V(|\varphi|^2) = 1 - |\varphi|^2 \tag{18}$$

equations (16), without noise, coincide with the Bogomolnyi equations [14] for vortices.

(5) Let us compute the exponential factor in the CMG (6) describing a change of the functional measure caused by the addition of a drift β. For $\varphi \in \mathbb{C}$ (equation (9)) we find

$$\frac{1}{2}\int \mathrm{d}t\, \mathrm{d}x \beta_\mu \beta^\mu = \int \overline{\partial_x \varphi} \partial_x \varphi \tag{19}$$

$$\int \mathrm{d}t\, \mathrm{d}x \beta^\mu e_{\mu a} \dot{b}^a = \mathrm{i} \int \mathrm{d}t\, \mathrm{d}x (\overline{\partial_x \varphi} \dot{b} - \partial_x \varphi \bar{\dot{b}}) = 2 \mathrm{Im} \int \mathrm{d}t\, \mathrm{d}x \partial_x \varphi (\overline{\partial_t \varphi} - V(\varphi)).$$

This is a surface term.

For the CP^n model we get

$$\frac{1}{2}\int \mathrm{d}t\, \mathrm{d}x \beta_\mu \beta^\mu = \int \mathrm{d}t\, \mathrm{d}x g_{\mu\bar{\nu}}(w) \partial_x w^\mu \overline{\partial_x w}^\nu$$

$$\int \mathrm{d}t\, \mathrm{d}x \beta^\mu e_{\mu a} \dot{b}^a = \int \mathrm{d}t\, \mathrm{d}x \beta^\mu e_{\mu a} e_{\nu a} \dot{\eta}^\nu$$

$$= \mathrm{i} \int \mathrm{d}t\, \mathrm{d}x g_{\mu\bar{\nu}}(w)(\partial_x w^\mu \overline{\partial_t w}^\nu - \partial_t w^\mu \overline{\partial_x w}^\nu) = Q. \tag{20}$$

Hence, the stochastic integral in equation (6) (the term with the time derivative) is equal to the *topological charge* [10], whereas the term β^2 adds to the Lagrangian in (5).

We find from equations (6), (17) and (18) analogous formulae for the Higgs model

$$L = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + \tfrac{1}{2}(\partial_\mu A_\mu)^2 + \overline{\nabla_\mu \varphi} \nabla_\mu \varphi + \tfrac{1}{2}(1 - |\varphi|^2)$$

$$Q = \int \tfrac{1}{2} F_{\mu\nu} \varepsilon_{\mu\nu} (1 - |\varphi|^2) - \varepsilon_{\mu\nu} \varepsilon_{ab} \nabla_\mu \varphi_a \nabla_\nu \varphi_b.$$

(6) The Markov property and invariance under the Euclidean group of the Euclidean fields are sufficient for a construction of relativistic quantum fields [15]. Our equations are Euclidean invariant and the Markov property is a direct consequence of the stochastic equations [7]. Hence, we expect that the stochastic equations determine a local relativistic quantum field. In order to describe this field in terms of a Lagrangian a relation between the stochastic equation and the functional measure has to be established. For a stochastic process this relation is determined by the CMG formula

(6). The stochastic equations (14)-(16) for random fields need regularisation to be well defined. After the regularisation the relation between the stochastic equation and the functional measure can be found from equations (7) and (8). Consider for example equation (14) on the lattice (with lattice spacing δ)

$$\partial_t^\delta \varphi + \mathrm{i}\partial_x^\delta \varphi - \overline{V(\varphi)} = \dot{b}$$

where ∂^δ is a lattice derivative and $\dot{b}$ are Gaussian random variables independent at every point of the lattice. Then, equation (7) is well defined on the lattice. If we change the variables from $\dot{b}$ to φ in equation (7), then we get the Yukawa model of [1] and [2] on the lattice. The spinor determinant will come out as the Jacobian of the transformation $\varphi \to \dot{b}$. We could also use a continuous time and the space lattice (or any other space regularisation). In such a regularisation the Hamiltonian (1) is well defined. Then, the stochastic equations (2) and (3) are treated by the standard probability theory [7]. The formula (6) becomes the well known Cameron-Martin-Girsanov formula with $\dot{b}\,\mathrm{d}t$ in the (regularised) topological charge being the Ito differential. In such a case the spinor determinant does not appear explicitly in the functional measure (but the stochastic Ito integral in equation (6) is in fact equal to the determinant in this regularisation; it is known [16] that the spinor determinant with free boundary conditions adds only a local term to the Lagrangian in quantum mechanics).

If we put equations (15) and (17) on a *spacetime* lattice, then we get also a spinor determinant in the functional measure in addition to the bosonic Lagrangian. From equation (15) we get det $\not{D}$, where D is the covariant derivative along the chiral field. In the Higgs model (17) there will be a spinor determinant describing the minimal coupling of the gauge field to a complex spinor and the Higgs field coupled to the complex as well as to a real spinor field. There remains to study the continuum limit of such a determinant in order to determine rigorously the functional measure corresponding to the stochastic equation in the continuum. This problem is now under investigation.

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